

Spin dynamics of hopping electrons in quantum wires: algebraic decay and noise

A.V. Shumilin,¹ E.Ya. Sherman,^{2,3} and M.M. Glazov^{1,4}

¹*Ioffe Institute, 194021 St.-Petersburg, Russia*

²*Department of Physical Chemistry, The University of the Basque Country, 48080 Bilbao, Spain*

³*IKERBASQUE Basque Foundation for Science, Bilbao, Spain*

⁴*Spin Optics Laboratory, St.-Petersburg State University,*

1 Ul'yanovskaya, 198504 Peterhof, St.-Petersburg, Russia

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We study theoretically spin decoherence and intrinsic spin noise in semiconductor quantum wires caused by an interplay of electron hopping between the localized states and the hyperfine interaction of electron and nuclear spins. At a sufficiently low density of localization sites the hopping rates have an exponentially broad distribution. It allows the description of the spin dynamics in terms of closely-situated “pairs” of sites and single “reaching” states, from which the series of hops result in the electron localized inside a “pair”. The developed analytical model and numerical simulations demonstrate disorder-dependent algebraic tails in the spin decay and power-law singularity-like features in the low-frequency part of the spin noise spectrum.

Introduction. Recent progress in semiconductor nanotechnology and request in new hardware elements for prospective quantum technology devices caused a strong interest in one-dimensional solid-state systems such as semiconductor nanowires and nanowire-based heterostructures. There are at least two main reasons for this interest. First reason is related to the abilities to produce and controllably manipulate electron spin qubits in InSb and InAs nanowires [1, 2] and superconductor-semiconductor-nanowire hybrids [3]. Second reason is that, InSb nanowires, due to a strong spin-orbit coupling, are important as the hosts of edge Majorana states in such hybrid structures [4–7]. These states, being of a fundamental interest for the quantum theory, are thought to be promising for the quantum computation applications as well. Disorder in the nanowires can play a crucial role in the physics of the qubits and Majorana states [8], and a variety of disorder regimes, dependent on the growth procedure and doping, is possible [9]. On the other hand, one-dimensionality leads to a strong localization of carriers, and the understanding of properties of localized electron states is requested for the understanding of these systems. In addition to the spin-orbit coupling, which can be reduced by choosing an appropriate nanowire realization and geometry, III-V semiconductors show a strong hyperfine interaction, that is a source of the spin dephasing of localized electrons [10, 11]. At a finite temperature, electrons can hop between different sites, and, therefore, experience randomly fluctuating hyperfine fields. Therefore, the hopping leads to spin relaxation and, thus, results in a spin noise. Recently, spin-noise spectroscopy experiments [12–14] have made it possible to access information about new regimes of spin dynamics, including a very long time evolution of electrons and nuclei [15–17]. The width of the spin-noise spectrum is related to the relaxation rate and the spectrum features seen as deviations from the Lorentzian shape can reveal various non-Markovian memory effects [18].

Here we address the spin relaxation and spin noise in semiconductor quantum wires due to the random hyper-

fine coupling induced by electron hopping between localized states. This disorder-determined relaxation is long and strongly non-exponential. The understanding of this spin relaxation mechanism can be valuable for the analysis of charge and spin transport in semiconductor nanowires and related hybrid structures.

Model. We consider a nanowire where the electrons are assumed to be localized at single donors or fluctuations of the wire width. The density of localization sites n and the localization length a satisfy the condition

$$na \ll 1, \quad (1)$$

meaning a weak overlap of the wavefunctions, Fig. 1. Electrons hop between the sites with the aid of acoustic phonons. At each localization site i the electron spin experiences a random static effective magnetic field with the precession frequency Ω_i . These frequencies are uncorrelated and isotropically distributed, $\langle \Omega_{i,\alpha} \Omega_{j,\beta} \rangle = \delta_{ij} \delta_{\alpha\beta} \Omega_0^2/3$, where $\alpha, \beta = x, y, z$ enumerate Cartesian components, the distribution of precession frequencies at a given site is Gaussian with the root-mean-square of Ω_0 [10, 19, 20]. Under the condition (1) the electron spin density matrix can be parametrized by the average occupancies, N_i , and spins, \mathbf{S}_i , at the sites. Here we consider small electron concentration, $N_i \ll 1$, neglect electron-electron interactions, and assume that each site is either empty or singly occupied [34]. The spin dynamics can be described by the set of kinetic equations [21, 22, 34]:

$$\frac{d\mathbf{S}_i}{dt} + \mathbf{S}_i \times \boldsymbol{\Omega}_i = \sum_j W_{ij} (\mathbf{S}_j - \mathbf{S}_i), \quad (2)$$

with W_{ij} being the hopping rates between the sites i and j . The spin-orbit interaction and hyperfine-irrelated spin-flip processes are disregarded. For the hopping rates we take the minimal model by assuming

$$W_{ij} = \frac{1}{\tau} \exp\left(-2 \frac{|x_i - x_j|}{a}\right), \quad (3)$$

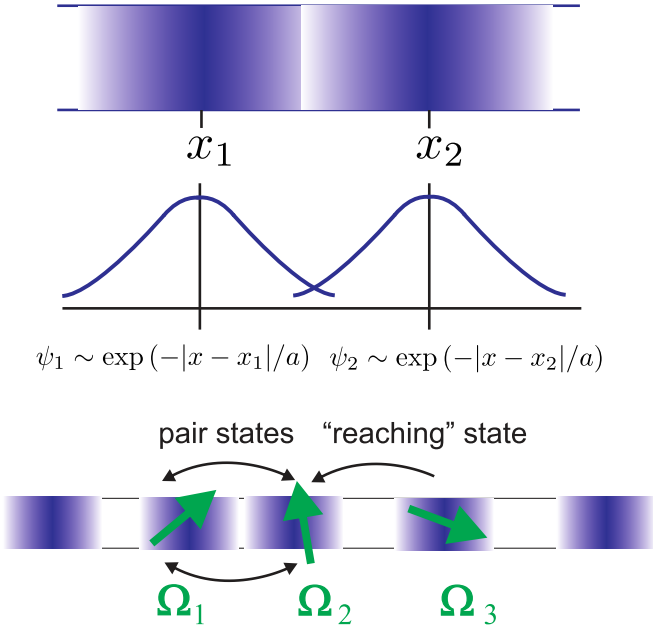


FIG. 1. Upper part: Sketch of a nanowire with localized states, schematically presenting the wavefunctions beneath the nanowire. Lower part: Illustration of a “pair” and a “reaching” state with hyperfine fields Ω_i (green arrows).

where x_i and x_j are the coordinates of the localization sites i and j , respectively, and τ is a prefactor governed by the strength of the electron-phonon interaction. The spread of a and τ is disregarded here [34]. Kinetic equation (2) enables us to evaluate the spin dynamics and spin fluctuations in the nanowire.

The condition (1) leads to an exponentially broad spread of the hopping rates. This effect together with possible multiple returns of hopping electrons to their initial sites determines the major specific features of the spin dynamics in nanowires. To analyze these phenomena in detail we further consider a realistic situation of rare transitions between the sites, $\Omega_0\tau \gg 1$, assuming, however, that the nuclear spin dynamics controlled by the hyperfine coupling with the electron spin, dipole-dipole or quadrupolar interactions is slow on the timescale of typical hops. Generally, the coupled electron-nuclear spin dynamics may by itself result in the non-exponential spin relaxation [23–26], in our case it results in the cut-off of the algebraic tail in the spin dynamics, see below. In the considered model, the electron spin at a given site i rapidly precesses in the static hyperfine field Ω_i , as a result, only its projection onto Ω_i is conserved during the typical hop waiting time [10, 27]. Then the electron hops to another site j and its spin precesses in the field Ω_j . Hence, the random hyperfine fields and hopping between the sites serve as a source of spin decoherence and relaxation. Specifics of spin decay in the opposite limit, $\Omega_0\tau \ll 1$, due to the Lévy distribution of the waiting times were studied in Ref. [28].

It is important that the condition $na \ll 1$, ensuring the

exponentially broad distribution of the hopping rates (3), leads to the situation where for the most of the sites the electron hop to its nearest neighbor is exponentially more probable than all other hops. It allows us to assume that the electron always hops to its nearest neighbor site and divide all the sites into two groups. For the first group of “pair” sites the relation “the nearest neighbor” is mutual, i.e. the two “pair” sites are the nearest neighbors for one another, see Fig. 1, cf. with the cluster model of Ref. [19]. The relaxation of the spin on “pair” sites is related to the multiple hops inside the pair. For the second group, the “reaching” sites, the “nearest neighbor” relation is not mutual. The “reaching” sites serve as the bottleneck for the spin relaxation: the longest waiting time in this situation is that of a hop for the site i to its nearest neighbor k . The subsequent relaxation is due to the hops to other sites that are closer to k than the site i and occur much faster than the initial hop $i \rightarrow k$. After several hops the electron reaches one of the “pair” sites and remains in this “pair”. Note, that in our model the typical distances between the sites in “pairs” is large compared with the localization radius a due to the condition (1). The spin dynamics and noise in a two-site complex with the distance $\lesssim a$ where the quantum tunnelling between the sites is important has been studied in Ref. [29].

Results. The contributions of the two groups of states to the long-time spin dynamics can be evaluated separately. The relaxation of the spin initially located on a “reaching” site i at times $t \gg \Omega_0^{-1}$ is determined by the rate of the fastest hop from this site to its neighbor. The expectation value of spin- z component [35] on the given “reaching” sites having the distance to the nearest neighbor close to r_i , can be estimated as $\langle s_{i,z}(t)/s_{i,z}(0) \rangle = s_R(t, r_i) = \exp(-t/\tau_i) K_{KT}(\Omega_0, t)$, where $\tau_i = \tau \exp(2r_i/a)$. Here $K_{KT} = \frac{1}{3} + \frac{2}{3}(1 - \Omega_0^2 t^2) \exp(-\Omega_0^2 t^2/2)$ is the Kubo-Tayabe formula [10, 27], which describes the spin precession in the static nuclear field Ω_i resulting in the depolarization of transverse to Ω_i spin components. Asymptotically (at $t \gg \Omega_0^{-1}$) K_{KT} tends to $1/3$. The electron spin localized on sites k and l forming a “pair” relaxes according a similar exponential law $s_P(t, r_{kl}, \theta_{kl}) = \exp(-t/\tau_{kl}) K_{KT}(\Omega_0, t)$. Here r_{kl} is the distance between the sites in the “pair” and θ_{kl} is the angle between the hyperfine fields at these sites, $\cos \theta_{kl} = (\Omega_k \cdot \Omega_l)/(\Omega_k \Omega_l)$ and time constant $\tau_{kl} = \tau \exp(2r_{kl}/a)/(1 - |\cos \theta_{kl}|)$ [21]. As a result, the disorder-averaged time dependence is given by

$$\left\langle \frac{S_z(t)}{S_z(0)} \right\rangle = \int_0^\infty P_R(r_i) s_R(t, r_i) dr_i + \frac{1}{2} \int_0^\infty dr_{kl} P_P(r_{kl}) \int_{-1}^1 d(\cos \theta_{kl}) s_P(t, r_{kl}, \theta_{kl}). \quad (4)$$

Here $P_R(r) = 2ne^{-2nr}(1 - e^{-nr})$ is the probability for a site to be a “reaching” one with the distance to its neighbor equal to r and $P_P(r) = 2ne^{-3nr}$ is a probability for a site to be a part of a “pair” with the distance between

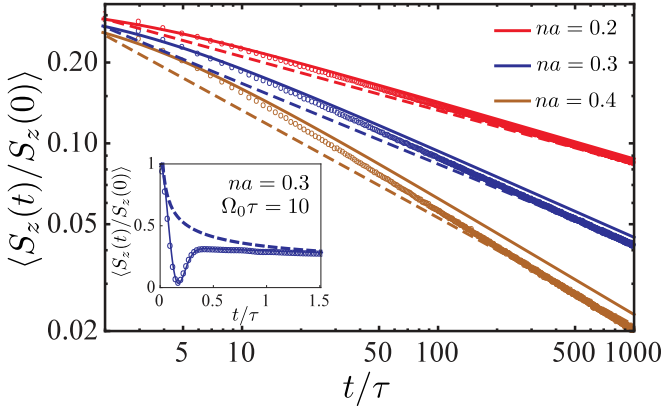


FIG. 2. The comparison of the numerical results (dots) with the model of “pair” and “reaching” sites, Eq (4), (solid lines) and power-law approximation Eq. (7) (dashed lines). Inset shows the comparison on the initial time interval of the relaxation for $na = 0.3$ and $\Omega_0\tau = 10$. Numerical calculation involved 100 sites and averaging over 5000 realizations of hyperfine fields.

sites r , and the Poisson distribution of sites is assumed. Equation (4) demonstrates that the total spin dynamics is given by the weighted average of the electron spins on the “reaching” and “pair” sites. The integrals in Eq. (4) can be expressed via special functions [34].

To get insight into the spin dynamics of hopping electrons in nanowires, we performed numerical simulations based on the kinetic Eqs. (2) and compared the results with the calculation after Eq. (4). Figure 2 shows very good agreement of the model calculation (solid lines) with numerics (dots) for $na \lesssim 0.4$. Note that the analytical model includes no free parameters.

Now we turn to the detailed analysis of the long-time, $t \gg \tau$, spin dynamics. Figure 2 indicates the power-law relaxation. To support this conjecture, we note that the spin at large $t \gg f\tau$, where the factor $f \gg 1$ [36], is governed by the sites with large distances to its neighbors, i.e. with large $r_i, r_{kl} \gg n^{-1}$ in Eq. (4). The sites with large distance to its nearest neighbor are typically the “reaching” sites because $P_R(r) \gg P_P(r)$ for $r \gg n^{-1}$. Indeed, let us consider 4 sites $(k, \dots, k+3)$ in a row. The sites $(k+1, k+2)$ form a pair if

$$x_{k+1} - x_k > x_{k+2} - x_{k+1} < x_{k+3} - x_{k+2}, \quad (5)$$

while the sufficient condition for a site, e.g., $k+2$ to be a “reaching” one is

$$x_{k+1} - x_k < x_{k+2} - x_{k+1} < x_{k+3} - x_{k+2}. \quad (6)$$

As a result, the distances in the “pairs” are statistically smaller than the distances between the “reaching” sites and their nearest neighbors. The “reaching” sites contribution into Eq. (4) can be analytically evaluated at $na \ll 1$ and yields the $t \gg \tau$ asymptotics [34]

$$\left\langle \frac{S_z(t)}{S_z(0)} \right\rangle = \frac{1}{3} C(na) \left(\frac{t}{\tau} \right)^{-na}, \quad (7)$$

where a coefficient $C(na) \sim 1$. Equation (7) clearly demonstrates that the long-time spin dynamics is described by the power law with the exponent controlled by the density of sites and the localization length. Since the “reaching” states serve as a bottleneck for the spin relaxation, the long-time spin evolution is controlled by the exponentially broad spread of the waiting times for these sites. The numerical simulations show that the asymptotic (7) with $C = 1$ is in a good agreement with the numerical results in a wide range of times and parameters na [34]. Only at a small $t \lesssim \tau$ the asymptotics (7) deviates from the numerical results, that agree with the general model, Eq. (4), see the inset in Fig. 2.

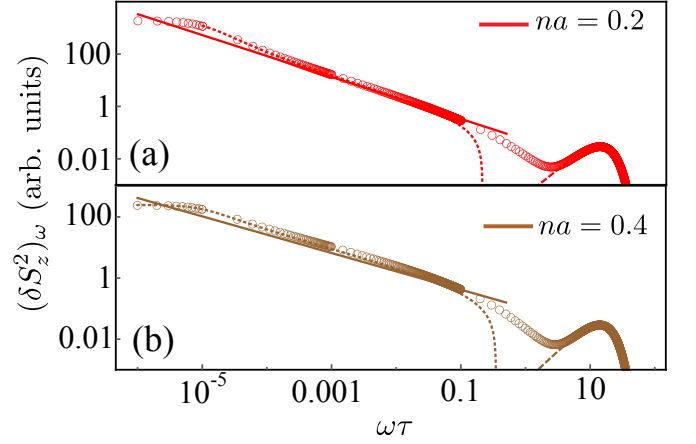


FIG. 3. Spin noise spectra calculated using Eqs. (4) and (8) (open circles), its power-law asymptotics, Eq. (9) (solid) and spin precession peak (dashed), $\Omega_0\tau = 10$. In the numerical calculation the exponential decay of spin polarization $\propto \exp(-t/\tau_s)$ with $\tau_s/\tau = 10^5$ was included. Dotted lines show asymptotics calculated with account for the spin relaxation time [34].

Spin noise. The power-law spin decay results in a zero-frequency anomaly in the spin noise spectrum. The autocorrelation function, $\langle \delta S_z(t) \delta S_z(t') \rangle$, obeys the same set of kinetic equations (2) [22, 30]. Defining the spin noise power spectrum in a standard way,

$$(\delta S_z^2)_\omega = \int \langle \delta S_z(t) \delta S_z(t') \rangle \exp(i\omega t) dt, \quad (8)$$

(see Refs. [20, 30–32] for details), we obtain by virtue of Eq. (7) the power-law feature in the spin noise spectrum [34]:

$$(\delta S_z^2)_\omega = \frac{\pi}{12} \frac{\tau na C(na)}{(\omega\tau)^{1-na}}, \quad \omega\tau \ll 1 \quad (9)$$

The ω^{na-1} feature in the spin noise spectrum is a direct consequence of the broad distribution of relaxation rates at the “reaching” sites. In general, the low-frequency spectrum is expected to be $1/\omega^\gamma$. In the above presented model, $\gamma = 1 - na$ and we obtain the “flicker” noise in the low-concentration limit. The exact value of γ depends

on the system details [34]. It is noteworthy that the singularity in the spin noise spectrum at $\omega = 0$ is smeared out due to nuclear spin dynamics: At the time scales on the order of the nuclear spin dephasing time, T_{2N} , caused, e.g., by the hyperfine coupling with the electron spin, the dipole-dipole coupling between neighboring nuclei or quadrupolar splittings, the frequencies Ω_i in Eq. (2) are no longer static resulting in the cut-off of the power-law feature in Eq. (9) at $\omega \lesssim 1/T_{2N}$.

Figure 3 shows the spin noise spectra calculated with Eqs. (4) and (8) in the model of “pairs” and “reaching” states for the two typical values of $na = 0.2$ and 0.4 . In the numerical calculation we have also introduced the exponential cut-off of the spin polarization replacing $K_{KT}(\Omega_0, t)$ in the expression for the spins at the “reaching” and “pair” states by the product $K_{KT}(\Omega_0, t) \exp(-t/\tau_s)$, where τ_s is the phenomenological spin relaxation time related, e.g., with the nuclear spin dephasing. Three frequency ranges are clearly visible in Fig. 3. At high frequencies $\omega \sim \Omega_0$ ($\Omega_0\tau = 10$ in our calculation) a peak in the spin noise spectrum is seen. It is related with the electron spin precession in the field of frozen nuclear fluctuation and serves as direct evidence of the hyperfine coupling of the electron and nuclear spins [17]. At this frequency range where $\omega \sim \Omega_0 \gg 1/(f\tau)$ the electron hopping is unimportant and the precession peak is described by the theory developed in Refs. [20, 22], see dashed lines in Fig. 3. In the range of intermediate frequencies $\tau_s^{-1} \ll \omega \lesssim (f\tau)^{-1}$ the spin noise spectrum is well described by a power-law asymptotics, Eq. (9), see solid lines in Fig. 3. This is a direct consequence of the algebraic tail in the spin relaxation, $S_z(t) \propto t^{-na}$, Eq. (7). Here, the power-law temporal decoherence of spins results in a power-law feature in the spin noise. Finally, at $\omega \lesssim \tau_s$ the algebraic singularity in the spin noise spectrum is suppressed and

$(S_z^2)_\omega$ reaches a constant value $\propto \tau_s$ at $\omega \rightarrow 0$. Overall, the ω -dependence of the spin noise power spectrum in the low-frequency domain is described by the dashed asymptotic presented in the Supplemental information [34], which takes into account the exponential spin relaxation at $t \gtrsim \tau_s$.

Conclusion. We have developed a theory of spin decoherence and corresponding spin noise in nanowires taking into account the key feature of disordered one-dimensional systems such as a strong electron localization resulting (i) in the electron hopping transport and (ii) enhanced hyperfine coupling between the electron and host lattice nuclear spins. As a result, the evolution of electron spin density fluctuations is governed by the interplay of the electron hopping and hyperfine interaction. Here the localization, at least for a low density of sites, results in the exponentially broad distribution of the hop waiting times and, consequently, in the algebraic long-time spin decay and in the power-law low-frequency singularity in the spin noise spectrum. It may be expected that the formation of the long-time spin decay due to exponentially wide distribution of waiting times is a general property of hopping electrons: In Ref. [33] it was shown for the spin relaxation dominated by the spin-orbit coupling.

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- [34] The role of the substantial occupancies, $N_i \sim 1$, the temperature-related effects and the proof of the robustness of the model are presented in the Supplemental Materials.
- [35] Since the considered system is isotropic in the spin space, the same results are valid for the relaxation of any of the spin components.
- [36] Formally, $f \sim \exp(2/na)$ which corresponds to the typical waiting time. However, the numerical analysis shows that the power-law asymptotics holds for substantially smaller times, see the Supplemental Materials for details.

Supplemental Material for “Spin dynamics of hopping electrons in quantum wires: algebraic decay and noise”

A.V. Shumilin,¹ E.Ya. Sherman,^{2,3} and M.M. Glazov^{1,4}

¹*Ioffe Institute, 194021 St.-Petersburg, Russia*

²*Department of Physical Chemistry, The University of the Basque Country, 48080 Bilbao, Spain*

³*IKERBASQUE Basque Foundation for Science, Bilbao, Spain*

⁴*Spin Optics Laboratory, St.-Petersburg State University,
1 Ul'anovskaya, 198504 Peterhof, St.-Petersburg, Russia*

S1. ROBUSTNESS TO THE DISTRIBUTION OF THE SYSTEM PARAMETERS

Here we prove robustness of our approach to the spread of the system parameters and to the effects of strongly temperature-dependent hopping probabilities. In addition, we will discuss the effects appearing due to a finite concentration of electrons where the effects related with the occupancy of states becomes important.

A. Robustness to the distribution of hopping matrix elements

In the main text we assumed the hopping rate probabilities for the states localized near the nanowire points x_i and x_j in the form, cf. Eq. (3),

$$W_{ij} = \frac{1}{\tau} \exp\left(-2\frac{|x_i - x_j|}{a}\right), \quad (S1)$$

were τ is determined by the electron-phonon coupling and a is the characteristic spread of the wavefunction along the wire axis. Here we estimate the typical for III-V semiconductors values of τ and show that the results in the main text are robust against the realistic spread of τ and a determined by a typical disorder in the system.

Localized wavefunctions $\psi_i(\mathbf{r})$ (with the corresponding eigenenergies E_i) are orthogonal to each other normalized eigenfunctions of the full Hamiltonian including the effect of the disorder. A typical spread of these energies is of the order of few meV. Electron hopping between the states with the energies E_i and E_j is accompanied by absorption or emission of a phonon with the momentum $q = |E_i - E_j|/\hbar c$ and occupation factor (at temperature $T \gg |E_i - E_j|$) of $n_q = T/|E_i - E_j|$. This momentum determines the electron-phonon coupling strength, the phonon density of states, and the wave-function dependent transition matrix elements given by the Fourier transform

$$M_{ij}(\mathbf{q}) = \int \psi_i^*(\mathbf{r}) \exp(\pm i\mathbf{q} \cdot \mathbf{r}) \psi_j(\mathbf{r}) d\mathbf{r}. \quad (S2)$$

In the $E_i - E_j = 0$ limit the matrix element $M_{ij}(\mathbf{q})$ vanishes due to orthogonality of the wave functions, and in the limit of large q it is small due to the fact that asymptotic of the Fourier transform of a regular function

should vanish at a large momentum. The large- q asymptotic strongly depends on the details of the shape of the wave functions. Accordingly, we model the hopping satisfying these conditions with the expression

$$W_{ij} = \frac{1}{\tau_D} \frac{\Delta^3 |E_i - E_j| T}{\Delta^5 + |E_i - E_j|^5} \times \left[\exp\left(-2\frac{|x_i - x_j|}{a_i}\right) + \exp\left(-2\frac{|x_i - x_j|}{a_j}\right) \right], \quad (S3)$$

vanishing at small and large $|E_i - E_j|$. Here a_i and a_j are the localization lengths for corresponding sites, and the parameter $\Delta \equiv \hbar c/a$, where a is the typical localization length and c is the longitudinal sound speed, determines the crossover between small and large momentum regimes, with the crossover momentum $q = 1/a$. For qualitative analysis we assume that electro-phonon coupling is due to the deformation potential, neglecting a relatively small contribution of piezointeraction [S1, S2]. According to the Fermi's golden rule, the characteristic hopping time determined by the electron-phonon coupling is estimated as $\tau_D^{-1} = D^2/\zeta a^3 \hbar c^2$. The values Δ and τ_D strongly depend on the system parameters. For the density $\zeta = 5 \text{ g/cm}^3$, $c = 5 \times 10^5 \text{ cm/s}$, and $a = 10 \text{ nm}$ we obtain $\Delta \approx 0.3 \text{ meV}$. The typical deformation potential $D = 7.5 \text{ eV}$ [S3, S4] yields $\tau_D \approx 10^{-2} \text{ ns}$. The hyperfine-induced spin precession rate Ω_0 is of the order of $\sim 10^9 \text{ s}^{-1}$. Bearing in mind that for $na < 0.4$, the exponent $\exp(2/na)$ is larger than 10^2 , this estimate implies that between the intersite jumps the electron spin experiences many rotations in the hyperfine field. It is worth mentioning that the slow algebraic decay of spin is observed already at times much smaller than $\tau \exp(2/na)$ (see footnote [36] in the main text and Sec. S2). Recalling that typical times for nuclear motion are of the order of or exceed $\sim 10^3 \Omega_0^{-1}$ [S5], we confirm the existence of the time interval where the electron hopping with long waiting times occurs while the nuclear hyperfine fields are still static.

To prove the robustness of our results we model the spin relaxation with the distribution of probabilities (S3) where the random energies are uniformly distributed in the interval $(E_0 - \Delta E/2, E_0 + \Delta E/2)$. To evaluate the corresponding wavefunction spread, we use the semiclassical asymptotic with $a_i = a_0 \sqrt{E_0/E_i}$, where a_0 corresponds to the E_0 energy.

The temperature and the energy cutoffs are taken

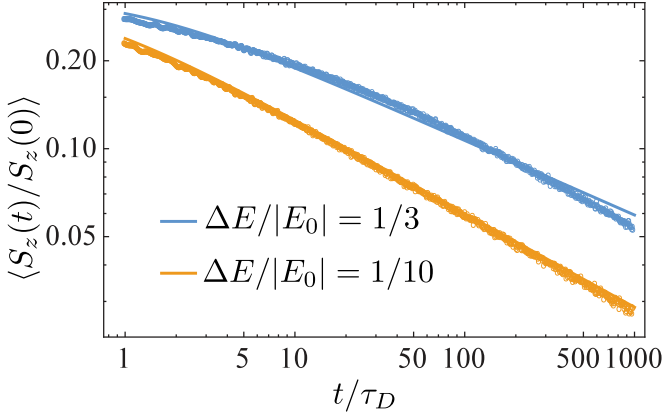


FIG. S1. The spin relaxation with finite temperature and random on-site energies. Points correspond to the numerical simulation based on Eq. (2) of the main text, lines correspond to the analytical model, Eq. (4) of the main text, with the parameters a and τ fitted to reproduce the results of numerical simulation.

as $T = |E_0|/3$ and $\Delta = 0.05|E_0|$. The results for $\Delta E = |E_0|/10$ and $\Delta E = |E_0|/3$ are shown in Fig. S1 and compared with Eq. (4) of the main text with the ΔE -dependent effective parameters τ and a . The parameters used in this comparison are $\tau = 0.25\tau_D$, $a = a_0$ for $\Delta E = 0.1|E_0|$ and $\tau = 0.6\tau_D$, $a = 0.8a_0$ for $\Delta E = |E_0|/3$. The results show that although the distribution of the pre-exponential factors in the hopping probabilities and energy-dependent localization length lead to some modification of the model parameters, the main features of the relaxation remain unchanged, proving the robustness of our model.

B. Effects of temperature and the electron concentration.

In the main text we focused on the system realization with a small mean occupation of localized states and the temperature considerably higher than their random energy spread, that is $T \gg |E_j - E_i|$. Here we discuss how the moderately low temperatures and finite electron concentrations modify the proposed picture.

The kinetic equations similar to Eq. (2) of the main text for finite electron concentrations read [S6]

$$\frac{d\mathbf{S}_i}{dt} + \mathbf{S}_i \times \boldsymbol{\Omega}_i = \sum_j [W_{ji}(1 - N_i)\mathbf{S}_j - W_{ij}(1 - N_i)\mathbf{S}_i], \quad (\text{S4})$$

where

$$N_i = \frac{1}{1 + \frac{1}{2} \exp\left(\frac{E_i - \mu}{T}\right)}$$

is the mean filling of the site j and μ is the chemical potential. Doubly occupied states are neglected. The

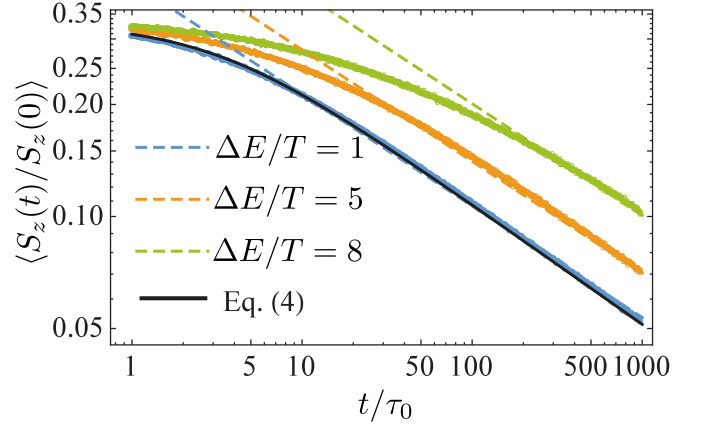


FIG. S2. The spin relaxation with finite temperature and random on-site energies. Points correspond to the numerical simulation based on eq. (S4), dashed lines correspond to the asymptotic A/t^{na} and the black line corresponds to the Eq. (4) from the main text.

hopping probabilities contain the temperature-dependent exponent as:

$$W_{ij} = \frac{1}{\tau_0} \exp\left(-\frac{|x_j - x_i|}{2a}\right) \times \left[\theta(E_i - E_j) + \exp\left(-\frac{|E_j - E_i|}{T}\right) \right]. \quad (\text{S5})$$

Here we consider the characteristic energy differences $|E_j - E_i|$ to be larger than the temperature. We also note that the derivation of (S4) disregards the correlations in the on-site density matrix and spin-spin exchange interaction - both effects can be important for sufficiently large occupations. Nevertheless, the Eqs. (S4) and (S5) can be used to qualitatively understand the suppression of the discussed relaxation and noise with decreasing temperature.

Figure S2 shows the numerical simulation based on Eqs. (S4) and (S5). The site energies were randomly chosen in the interval $(\mu - 0.1\Delta E, \mu + 0.9\Delta E)$ near the chemical potential level. It ensures that the electron concentration n_e is relatively small compared to the site concentration n : $n_e/n \approx 0.18$ for the $\Delta E/T = 8$ and $n_e/n \approx 0.5$ for the $\Delta E/T = 1$. The spatial distribution of the sites is Poissonian with $na = 0.3$. The results are compared with the results of Eq. (4) from the main text and with the power-law asymptotic A/t^{na} .

For relatively small $\Delta E \sim T$ the calculations reproduce very well the relaxation considered in the main text. They agree with Eq. (4) from the main text with parameter $\tau = 1.5\tau_0$. However when ΔE is large the initial part of relaxation at times $t \sim \tau$ is suppressed. This relaxation is related to the pairs of sites that are separated by a distance $r \lesssim a$ and both have the energy near the Fermi level. Such pairs become rare when $\Delta E \gg T$. At sufficiently large times the relaxation behaves as $S_z(t) = A/t^{na}$ with the power na insensitive

to ΔE .

The independence of the power law asymptotic of the random energies is related to the sharp cutoff of the energies at the interval boundaries. When the exponential band tails are included into consideration, the algebraic relaxation with a temperature-dependent power appears even without a spatial disorder [S7].

S2. ASYMPTOTES OF SPIN DECAY AND OF SPIN NOISE

Equation (4) of the main text can be presented in the form:

$$\left\langle \frac{S_z(t)}{S_z(0)} \right\rangle = \left\langle \frac{S_z(t)}{S_z(0)} \right\rangle_r + \left\langle \frac{S_z(t)}{S_z(0)} \right\rangle_p, \quad (\text{S6})$$

$$\left\langle \frac{S_z(t)}{S_z(0)} \right\rangle_r = \frac{na}{3} \left(\frac{t}{\tau} \right)^{-3na/2} \left\{ \left(\frac{t}{\tau} \right)^{na/2} [\Gamma(na) - \Gamma(na, t/\tau)] + \Gamma\left(\frac{3na}{2}, \frac{t}{\tau}\right) - \Gamma\left(\frac{3na}{2}\right) \right\}, \quad (\text{S7})$$

where $\Gamma(x)$ is the Γ -function and $\Gamma(x, y)$ is the incomplete Γ -function defined as

$$\Gamma(x, y) = \int_y^\infty u^{x-1} e^{-u} du.$$

The contribution of pairs reads

$$\left\langle \frac{S_z(t)}{S_z(0)} \right\rangle_p = \frac{na\tau}{3t} \left[E_\nu(t/\tau) + \frac{2}{3na-2} \right] - \frac{na}{3\tau} t^{-3na/2} \Gamma\left(\frac{3na}{2} - 1\right), \quad (\text{S8})$$

where

$$\nu = 2 - \frac{3na}{2},$$

and the generalized exponential integral function

$$E_\nu(x) = \int_1^\infty \frac{e^{-xu}}{u^\nu} du = x^{\nu-1} \Gamma(1-\nu, x),$$

are introduced.

Linear- t asymptotics (valid at the short times, formally $1/\Omega_0 \ll t \ll \tau$):

$$\left\langle \frac{S_z(t)}{S_z(0)} \right\rangle = \frac{1}{3} - \frac{na(1+2na)t}{3(1+na)(2+3na)}. \quad (\text{S9})$$

At long times, $t \gg \tau \exp(2/na)$ we have

$$\left\langle \frac{S_z(t)}{S_z(0)} \right\rangle = \frac{\Gamma(na)}{3} \left(\frac{t}{\tau} \right)^{-na}. \quad (\text{S10})$$

This leading order asymptotics results from the reaching states only, Eq. (S7). At $na \rightarrow 0$ the $\Gamma(na) \rightarrow 1$ and Eq. (S10) is in agreement with Eq. (7) of the main text.

with

$$\left\langle \frac{S_z(t)}{S_z(0)} \right\rangle_r = \int_0^\infty P_R(r_i) s_R(t, r_i) dr_i,$$

and

$$\left\langle \frac{S_z(t)}{S_z(0)} \right\rangle_p = \frac{1}{2} \int_0^\infty dr_{kl} P_P(r_{kl}) \int_{-1}^1 d(\cos \theta_{kl}) s_P(t, r_{kl}, \theta_{kl}).$$

These integrals can be evaluated analytically at $t \gg 1/\Omega_0$ via special functions. For the reaching states contribution one has:

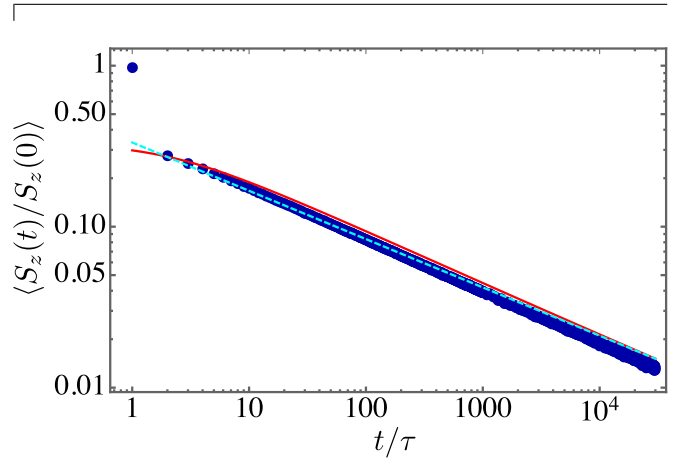


FIG. S3. Numerical simulation of the long-time behavior of the spin decay (circles) for $na = 0.3$ calculated in the model of the main text. Dashed cyan line shows the asymptotics presented in Eq. (7) of the main text with $C = 1$ and red line shows the analytical result, Eqs. (S6), (S7), and (S8). We used 100 sites and averaged the result over 1700 realizations.

Figure S3 shows the comparison of the numerical simulation (points) with the analytical formulae Eqs. (S6),

(S7), and (S8) (red line) and asymptotics (7) of the main text. Overall, a good agreement is observed.

Spin noise spectrum is defined by [cf. Eq. (8) of the main text]

$$\begin{aligned} (S_z^2)_\omega &= \int \langle S_z(t+t') S_z(t) \rangle \exp(i\omega t') dt' \\ &= \frac{1}{4} \int \left\langle \frac{S_z(t')}{S_z(0)} \right\rangle \exp(i\omega t') dt', \end{aligned} \quad (\text{S11})$$

where the factor 1/4 in the second equality takes into account the fact that the single-time correlator of spin

components equals to $1/4 = (1/3) \cdot 1/2 \cdot (1/2 + 1)$. Taking $\langle S_z(t)/S_z(0) \rangle$ in the form of Eq. (7) of the main text multiplied by $\exp(-t/\tau_s)$ where τ_s is the phenomenological spin relaxation time (cut-off time, related, e.g, with nuclear spin dynamics) we get in the limit of $\omega \ll 1/\tau$:

$$\begin{aligned} (S_z^2)_\omega &= \frac{C(na)}{6} \text{Re} \left\{ E_{na} \left(\frac{\tau}{\tau_s} - i\omega\tau \right) \right\} \\ &\approx \frac{C(na)}{6} \left(\frac{\tau}{\tau_s} \right)^{na-1} \text{Re}\{(1 - i\omega\tau_s)^{na-1}\}. \end{aligned} \quad (\text{S12})$$

This expression passes to Eq. (9) of the main text at $\tau_s \rightarrow \infty$.

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